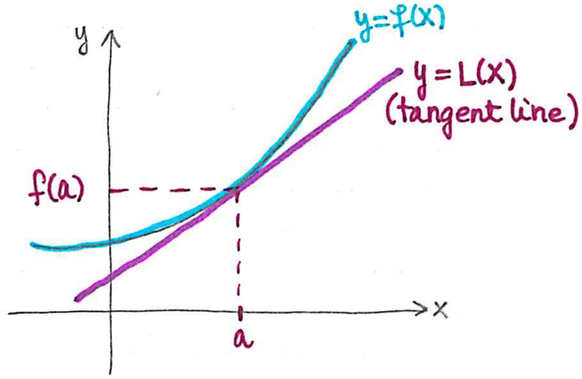
2.9 Linear Approximation & Differentials.



Equation of tangent line:

$$y = f(a) + f'(a)(x-a)$$

Linear Approximation of f at a:

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

The Linearization of f at a:

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

Example 1: $f(x) = 9x^{-3}$

ⓐ Linearization of f at $x=3$?

$$f'(x) = -9 \cdot 3x^{-4} = -27x^{-4}$$

$$\Rightarrow f'(3) = -27\frac{1}{3^4} = -\frac{1}{3} \quad ; \quad f(3) = 9 \cdot \frac{1}{3^3} = \frac{1}{3}$$

$$\boxed{L(x) = \frac{1}{3} - \frac{1}{3}(x-3)}$$

ⓑ Use the linearization to estimate $f\left(\frac{14}{5}\right)$:

$$L\left(\frac{14}{5}\right) = \frac{1}{3} - \frac{1}{3}\left(\frac{14}{5} - 3\right)$$

Example 2: What is the best function $f(x)$ and value a so that the linearization of $f(x)$ at $x=a$ can be used to estimate $\sqrt{24.8}$?

$f(x) = \sqrt{x}$; at $a = 25$.

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(25) = \frac{1}{10} \Rightarrow \boxed{L(x) = 5 + \frac{1}{10}(x-25)}$$

$$L(24.8) = 5 + \frac{1}{10}(-0.2) = 5 - \frac{2}{100} = 5 - \frac{1}{50}$$

Example 3: Use a linear approx. to estimate $\sin(28°)$.

$$f(x) = \sin(x); \quad a = 30° = \frac{\pi}{6}$$

$$f'(x) = \cos(x); \quad f'(\pi/6) = \sqrt{3}/2$$

$$L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$$

180° ____ π

28° ____ ?

$? = \frac{28\pi}{180}$

$$\Rightarrow L(28°) = L\left(\frac{28\pi}{180}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{28\pi}{180} - \frac{\pi}{6}\right) = \frac{1}{2} - \frac{\sqrt{3}}{\cancel{2}} \cdot \frac{\cancel{2}\pi}{180} = \frac{1}{2} - \frac{\sqrt{3}\pi}{180}$$

$\frac{30\pi}{180}$

Differentials

If $y = f(x)$, the differentials dy and dx are defined by:

$$dy = f'(x)dx$$

Example: $y = x^2 - 4x + \frac{1}{x}$

$$dy = \left(2x - 4 - \frac{1}{x^2}\right) dx.$$

If Δx is the change in x, then $\Delta y = f(x+\Delta x) - f(x)$ is the corresponding change in y. When Δx is small, we often assume $\Delta y \approx dy$.

Example: Use differentials to estimate the change in $y = \frac{3}{x^2}$ as x moves from 3 to 3.1.

$$y = \frac{3}{x^2}$$

$$dy = -\frac{3.2}{x^3} dx = -\frac{6}{x^3} dx$$

Change from 3 to 3.1: Put $x = 3$ and $dx = \Delta x = (3.1) - 3 = 0.1$

$$\Rightarrow dy = -\frac{6}{3^3} \cdot (0.1) = -\frac{2}{9} \cdot \frac{1}{10} = \boxed{-\frac{2}{90}}$$

Example: Radius of a sphere: measured to be 8 cm w/ a possible error of $\frac{1}{4}$ cm.

(a) Maximum error in the calculated surface area?

$$S = 4\pi R^2$$

Denote error in R by $\Delta R \cong dR = \frac{1}{4}$

$$dS = 4\pi \cdot 2R\, dR$$

$$\Rightarrow dS = \cancel{4}\pi \cdot 2 \cdot 8 \cdot \frac{1}{\cancel{4}} \Rightarrow \boxed{dS = 16\pi}$$

(b) Error in calculated volume?

$$V = \frac{4}{3}\pi R^3$$

$$dV = \frac{4}{3}\pi \cdot 3R^2 dR$$

$$\Rightarrow dV = \frac{\cancel{4}}{\cancel{3}}\pi \cdot \cancel{3} \cdot 8^2 \cdot \frac{1}{\cancel{4}} = \boxed{64\pi}$$